Use of Shear Lag for Composite Microstress Analysis—Linear Array

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The task of finding stress distributions in damaged composites analytically by means of the theory of elasticity is prohibitively difficult, so approximate techniques are generally employed. The most widely used is shear lag theory. This theory contains a parameter representing the shear coupling between fibers that has been difficult to evaluate in the past. The present paper evaluates this parameter for a linear array of fibers. In the limiting case of small fiber diameter to fiber spacing ratio and large matrix thickness to fiber diameter ratio an analytical solution is possible and is provided in this paper. For other geometries the solution is obtained experimentally. It is found that the assumption usually made for the parameter is rather far off for most geometries of practical interest.

Nomenclature

d	= distance between fiber or rod centers
d_1	= distance between next-to-nearest fibers in rectangular array
d_2	= distance between nearest fibers in rectangular array
D_0	= large distance approaching infinity at which the displacement of the field point is essentially zero
F_i'	= load transfer to the ith fiber per unit length
F_0'	= axial line force per unit length
G	= shear modulus of the matrix
Gh	= effective shear stiffness of the composite
h/d	= parameter appearing in shear lag equation for fiber interaction
$(h/d)_{c,v}$	= h/d for the interaction between nearest fiber pair in a rectangular array
I	= current generated by the power supply
r_0	= radius of a fiber or a rod
R	= resistance
t	= lamina thickness (thickness of matrix layer)
w_i	= axial displacement of the ith fiber
z	= axial coordinate
α	= ratio of the distance between next-to-nearest fiber pair to the distance between nearest fiber pair in a rectangular array
ρ_e	= resistivity of the electrolyte
ϕ	= electrical potential

Introduction

STUDIES of the micromechanics of failure of unidirectionally reinforced composites are based on theoretical analyses of the stress distribution in damaged composites under load. Any use of the theory of elasticity for this purpose turns out to be prohibitively difficult, and therefore approximate techniques must be employed. A number of authors have used shear lag theory or some modification thereof for this purpose.[1-5] However, the relation between the solutions found in this manner and experimental results has been made uncertain by the fact that the solutions involve a parameter the values of which were not really known. This parameter is the effective shear stiffness of the matrix, which serves to transfer direct stress from one fiber to another.

Several previous papers deal with various aspects of this question. One gives an exact solution for the shear interaction between two infinitely long rigid fibers immersed in an infinite elastic medium.[6] Another gives an approximate solution for the interaction between neighboring pairs of fibers in a square array.[7] A third extends this solution to the case of a rectangular array.[8]

The present paper considers a linear array, i.e., a unidirectionally reinforced lamina or tape. A theoretical solution is given for such an array that is valid when the fiber diameter is small compared to fiber separation, and fiber separation is small compared to matrix thickness. For other geometries, the solution is obtained experimentally using an electric analog technique.[8] These findings are compared to assumptions that have been made by previous investigators concerning the shear transfer stiffness.

Analytical Solution for Linear Array

Shear lag was originally devised to simplify the stress analysis of stiffened sheet construction in aircraft. The equations take the form

$$\frac{\mathrm{d}F_i(z)}{\mathrm{d}z} = G\left(\frac{h}{d}\right)(-w_{i-1}+2w_i-w_{i+1}) \tag{1}$$

where d is the distance between stiffeners, h the sheet thickness, G the shear modulus of sheet material, $F_i(z)$ the load in the ith stiffener, and w_i the axial displacement in the ith stiffener.

It was initially assumed by Hedgepeth,[1] and later by numerous others, that the same equation applies to uniaxially reinforced continuous fiber composites. However, when the matrix layer thickness is appreciably larger than the fiber diameter, h loses its simple physical significance and requires careful evaluation.

We start the evaluation of h by considering three infinitely long rigid fibers immersed in an infinite elastic medium with static equilibrium maintained, as shown in Fig. 1. The center fiber is loaded with an axial line force $2F_0'$ per unit length, while the neighboring fibers are each loaded with F_0' directed in the opposite sense.

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The shear lag equation for the center fiber becomes

$$\frac{dF_a}{dz} = F_a' = G\left(\frac{h}{d}\right)(-w_c + 2w_a - w_b) = 2F_0' \qquad (2)$$

Since $w_b = w_c$

$$F_a' = 2G(h/d)(w_a - w_b) \qquad (3)$$

Reference 6 considered an infinitely long rigid rod of radius r_0 immersed in a very large concentric elastic cylinder of radius D_0. It was shown that if an axial force per unit length F_0' is applied to the rod and the cylinder periphery remains fixed, the rod displacement becomes

$$w(r_0) = -(F_0'/2\pi G)\ln(r_0/D_0) \qquad (4)$$

The elastic material in the cylinder is displaced a distance $w(r)$ where

$$w(r) = -(F_0'/2\pi G)\ln(r/D_0) \qquad (5)$$

The displacement of each fiber in the setup of Fig. 1 can be obtained by superposing the effects of each of the applied loads. Employing Eqs. (4) and (5),

$$w_a = -\frac{2F_0'}{2\pi G}\ln\left(\frac{r_0}{D_0}\right) + \frac{2F_0'}{2\pi G}\ln\left(\frac{d}{D_0}\right)$$

$$= \frac{2F_0'}{2\pi G}\ln\left(\frac{d}{r_0}\right) \qquad (6)$$

and

$$w_b = -\frac{F_b'}{2\pi G}\ln\left(\frac{r_0}{D_0}\right) - \frac{F_a'}{2\pi G}\ln\left(\frac{d}{D_0}\right)$$

$$-\frac{F_c'}{2\pi G}\ln\left(\frac{2d}{D_0}\right) = \frac{F_0'}{2\pi G}\ln\left(\frac{2r_0}{d}\right) \qquad (7)$$

Subtracting Eq. (7) from Eq. (6), we obtain

$$w_a - w_b = \frac{F_0'}{2\pi G}\left[2\ln\left(\frac{d}{r_0}\right) + \ln\left(\frac{d}{2r_0}\right)\right] \qquad (8)$$

Thus,

$$2(w_a - w_b) = \frac{F_a'}{2\pi G}\left[3\ln\left(\frac{d}{r_0}\right) - \ln 2\right] \qquad (9)$$

or

$$F_a' = \frac{2\pi[2G(w_a - w_b)]}{[3\ln(d/r_0) - \ln 2]} \qquad (10)$$

Comparing Eq. (10) with Eq. (3),

$$\frac{h}{d} = \frac{2\pi}{3\ln(d/r_0) - \ln 2} \qquad (11)$$

Experimental Setup

The theory underlying the electric analog can be found in Ref. 8. A linear array with various thicknesses of matrix material was simulated by placing two vertical insulating panels with height exceeding the depth of the fluid (electrolyte), on top of a submerged panel with holes that match the size of the conducting rods and with spacing appropriate to the r_0/d under investigation. The insulating panels were

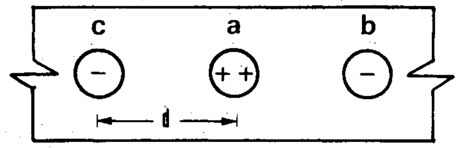

Fig. 1 A three-fiber interaction in which static equilibrium is maintained.

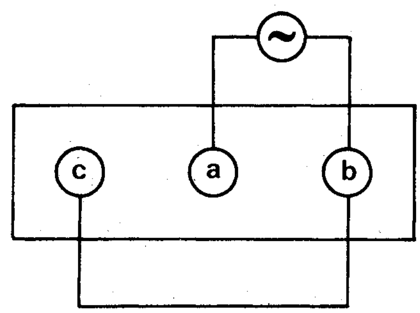

Fig. 2 Simulation of a three-fiber interaction by an electric analog.

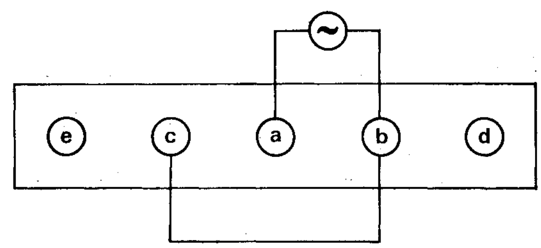

Fig. 3 Simulation of a five-fiber interaction by an electric analog.

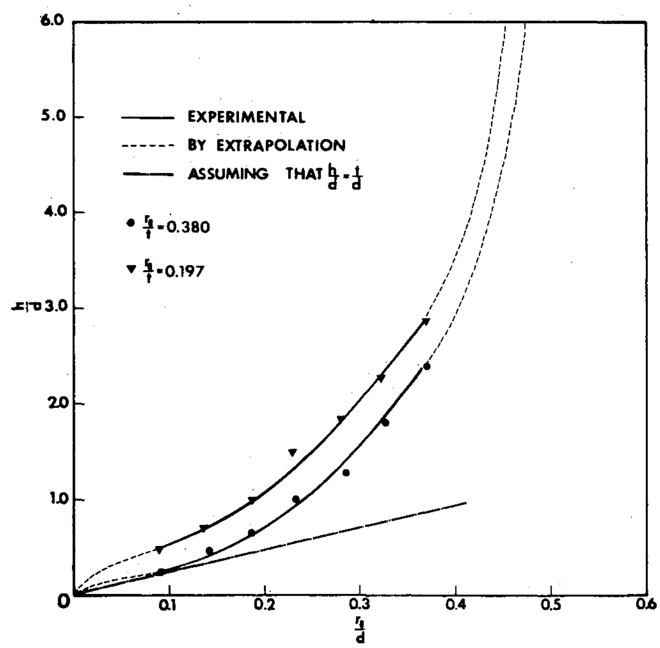

Fig. 4 h/d for linear array of fibers, $r_0/t = 0.380$ and 0.197.

Table 1 Comparison of a linear array with a very rectangular array with same ratio of fiber radius to fiber spacing

(r_0/d), (r_0/d_2)	0.370	0.325	0.280	0.230
h/d	3.477	2.724	2.263	1.857
$(h/d)_{c,v}$	3.535	2.712	2.214	1.830

spaced at equal distances from the centerlines of the rods. The distance was chosen to represent the desired composite thickness. Thus, the fluid between the vertical panels was analogous to the matrix associated with that particular array.

Linear arrays with differing thicknesses and rod spacing were investigated, the first experiment involving three rods, as in Fig. 2. A constant ac potential was applied across rods a and b, with b connected to c. The current through the power supply was measured in each case. The potential difference between a and b was also measured. The governing equation can be written as

$$\frac{\mathrm{d}I}{\mathrm{d}z}=\frac{1}{\rho_e}\left(\frac{h}{d}\right)(2\phi_{\mathrm{ab}}) \tag{12}$$

The effect of the presence of a free rod adjacent to b and c was investigated by a second experiment, as in Fig. 3.

Results

The results of the experiments for evaluating h/d are summarized in Figs. 4-6. In these figures h/d is plotted against r_0/d for various ratios of fiber radius to lamina thickness, r_0/t. We do not have a theoretical solution except for $r_0/t \to 0$, so Figs. 4-6 are based entirely on the experimental data. Extrapolation beyond the data points is aided by the knowledge that $h/d \to 0$ for $r_0/d \to 0$, and $h/d \to \infty$ when $r_0/d \to 0.5$. These limiting cases are discussed in Refs. 7 and 8.

Figure 7 shows the results for h/d for a linear array of rods in an open tank (i.e., the vertical panels are removed, so the matrix has the shape of the tank). Also shown are the theoretical results for an infinite tank. We note that, as expected, the theoretical results agree very closely with the experimental data obtained in a large but finite tank for small values of r_0/d, i.e., where the theoretical solution is valid.

In the case of a square or rectangular array, four neighboring fibers must be considered. Another limiting case is obtained by noting that for a very rectangular array ($d_1/d_2=\alpha \gg 1$) the nearest fiber interaction should be almost the same as though the adjacent lamina were not there. Table 1 gives the h/d obtained for $\alpha=10.5$ and shows that the results are substantially the same as those obtained for a linear array in an open tank.

In this investigation, most of the data were obtained using three rods, as shown in Fig. 1. A linear array is better approximated by a very large number of rods, but a very large number of rods will in many cases exceed the capacity of the tank. To see whether additional rods are really needed, the effect of adding an additional rod at each end, as in Fig. 3, was investigated. The added rods were left free to adopt the local potential. The results are contained in Table 2. It was found that the resistance was virtually unaffected by the presence of the additional rods. The largest change was a few percentage points, and it occurred when r_0/d and r_0/t were both large. If either was small, the change in resistance was negligible. We conclude that the results obtained using three rods are sufficiently accurate for most engineering purposes.

Discussion

Reedy[9] has evaluated shear lag by comparing its stress predictions with those of a three-dimensional finite element

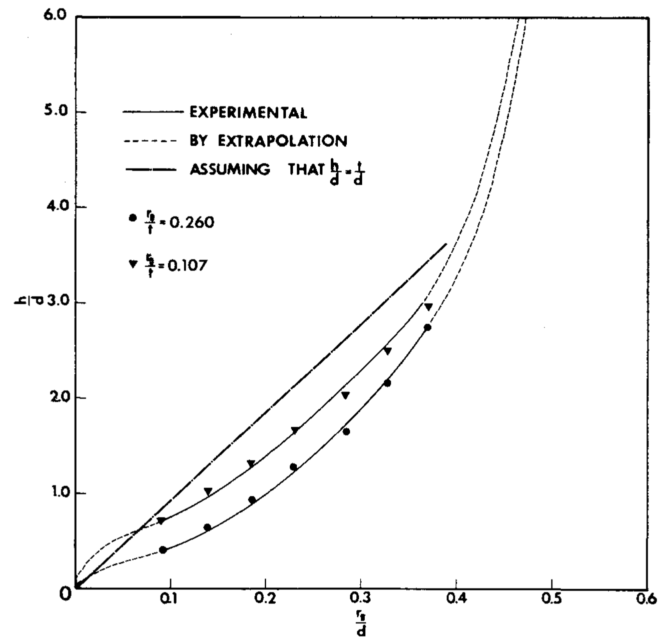

Fig. 6 h/d **for linear array of fibers,** $r_0/t=0.260$ **and 0.107.**

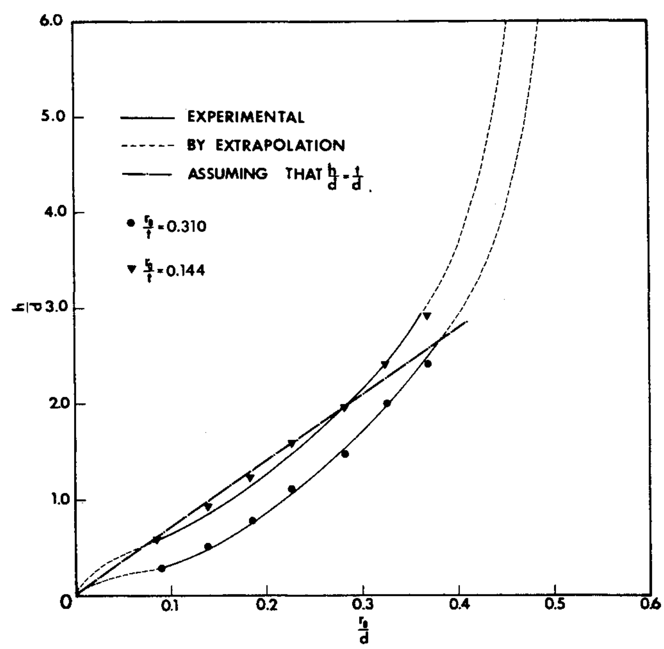

Fig. 5 h/d **for linear array of fibers,** $r_0/t=0.310$ **and 0.144.**

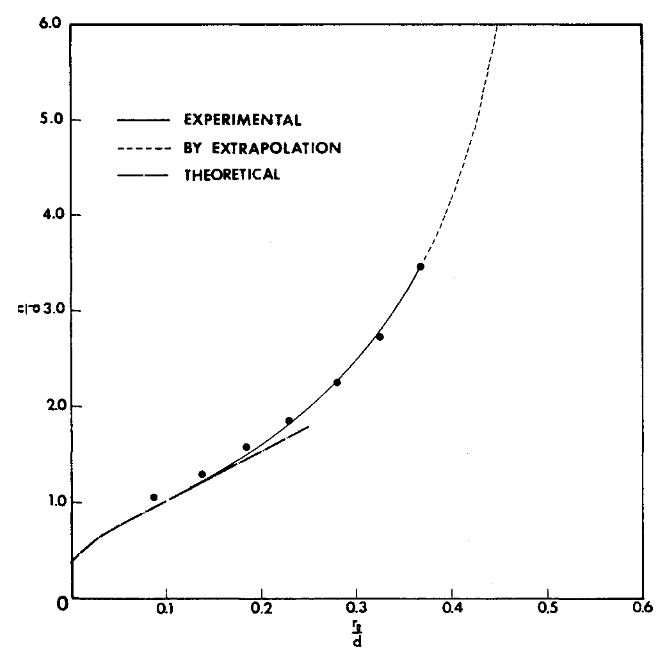

Fig. 7 h/d **for linear array of fibers (open tank).**

Table 2 Resistance due to three- and five-fiber interaction

	r_0/t					
r_0/d	0.380	0.310	0.260	0.197	0.144	0.107
0.370						
R_3[a]	98.40	97.77	86.14	83.17	81.14	79.67
R_5[b]	98.90	97.28	86.81	85.30	81.14	79.67
0.325						
R_3	132.67	119.64	109.23	103.23	98.12	95.63
R_5	134.32	121.23	113.07	102.16	97.92	95.73
0.280						
R_3	193.08	169.49	149.55	135.54	127.56	122.84
R_5	192.88	171.28	149.55	135.41	127.56	122.98
0.230						
R_3	248.50	222.00	196.08	169.49	157.14	150.00
R_5	250.25	222.22	195.88	169.49	157.30	150.00
0.186						
R_3	369.63	321.61	269.19	231.63	199.40	187.55
R_5	369.63	321.61	268.92	231.63	199.00	187.74
0.140						
R_3	526.32	475.24	398.80	331.67	261.58	236.67
R_5	526.32	475.24	399.20	331.67	261.58	236.67
0.090						
R_3	993.00	827.50	621.25	496.50	413.75	342.07
R_5	993.00	827.50	621.25	496.50	413.75	342.07

[a]Resistance of three-fiber interaction (ohms). [b]Resistance of five-fiber interaction (ohms).

calculation. To accomplish this he made plausible engineering approximations that led to values of h/d in good agreement with those obtained here for low and intermediate values of the fiber volume ratio, but poor agreement for high values of fiber volume ratio. Other investigators who have employed shear lag to find stress distributions in single-ply composites have assumed that $h/d = t/d$, irrespective of the value of r_0/d. Since $t/d = (t/r_0)(r_0/d)$, a plot of t/d vs r_0/d is a straight line of slope t/r_0. Such lines are shown in dot-dash format for $r_0/t = 0.107$, 0.144, and 0.380 in Figs. 4-6. We see that the assumption that $h/d = t/d$ is not bad for most of the r_0/d range when $r_0/t = 0.144$, but is rather far off for values of r_0/t differing widely from this value.

This paper and its predecessors[7,8] have attempted to find the shear interaction to be used under the simplifying assumption generally employed in shear lag theory that only nearest-neighbor fibers interact. The accuracy of this basic assumption is currently unknown.

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